Log abundance for Kähler threefolds

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Complex Varieties

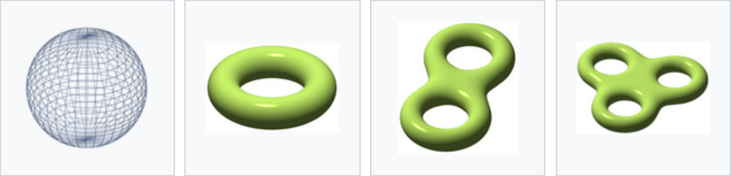
- A complex manifold X is a manifold with an atlas of charts to the open balls in $\mathbb{C}^n$, such that the transition maps are holomorphic. We denote by Ω^1_X the holomorphic cotangent bundle and by $K_X = \Omega^n_X$ the canonical line bundle.
- Example: affine spaces $\mathbb{C}^n$, projective spaces $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}$, tori $\mathbb{C}^n/\mathbb{Z}^{2n}$
- A standard complex analytic space V is a subspace of a domain $U \subseteq \mathbb{C}^N$ such that $V \subseteq U$ is defined by finitely many holomorphic functions.
- More generally, a complex analytic space is a ringed space which is locally isomorphic standard analytic spaces. A complex variety is an integral analytic space. It is a complex manifold if and only if it is smooth.
- Assume that X is a compact complex variety, then there is a bimeromorphic morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ is a complex manifold.

Complex Varieties

- A compact complex variety X is called a *Kähler* variety if it carries a Kähler form, that is, a closed definite positive $(1,1)$-form (Kähler 1933).
- A complex variety X is called projective if it is an analytic subvariety of the projective space $\mathbb{P}^m$.
- Every complex projective variety X is compact Kähler. Chow's theorem (1949) also asserts that X is then defined globally as the zero locus of finitely many homogeneous polynomials. Serre's GAGA principle (1956) then allows us to use algebraic methods to study projective complex varieties.
- A complex compact variety X is projective if and only if it carries an *ample* line bundle L, that is, $c_1(L) \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Q})$ is a Kähler class (Kodaira 1954).
- Goal: Classify (compact) complex manifolds.

Classification of Curves

- Every compact complex curve is projective. Smooth irreducible curves C can be classified according to their genera $g(C)$. More roughly, we can divide in three classes as follows.
- $g(C) = 0$, $C \cong \mathbb{P}^1$ is called a rational curve.
- $g(C) = 1$, C is called an elliptic curve.
- $g(C) > 1$, C is a higher genus curves.



Genre 0. Genre 1. Genre 2. Genre 3.

Classification of Surfaces

- Classification of surfaces was already initiated by the Italian school in the late 19th century (Albanese, Bertini, Castelnuovo, del Pezzo, Enriques, Segre, Severi...).
- Starting from dimension 2, there is a new type of operation, blow-up of a point (subvariety).

successive blow-ups of three points

Classification of Surfaces

- It is natural to consider X and its blow-ups in the same class. Indeed, we can "inverse" a blow-up by the following theorem.

Theorem (Castelnuovo)

Assume that $C \cong \mathbb{P}^1$ is a rational curve in a surface Y such that $C^2 = -1$ (Such a curve is called a (-1)-curve). Then there is a morphism $Y \to X$ blowing down C to a point.

- Starting from a compact complex surface X, we can blow down successfully (-1)-curves. After *finitely many* steps, we arrive at a surface S which contains no (-1)-curve. Such a surface is called a *minimal surface*.

- A minimal surface is not the blow-up of any surface.

Classification of Surfaces

- Projective minimal surfaces are classified by Enriques (1910s) and non algebraic minimal compact surfaces are classified by Kodaira (1960s)
- For every compact complex manifold, the *Kodaira dimension* $\kappa(X)$ of X is the transcendental degree over $\mathbb{C}$ minus 1 of the pluricanonical section ring

$$R(K_X) = \bigoplus_{m=0}^{+\infty} H^0(X, K_X^{\otimes m}) = \bigoplus_{m=0}^{+\infty} H^0(X, mK_X).$$

  We have $\kappa(X) \in \{-1, 0, ..., \dim X\}$.

- According to the Kodaira dimensions, we have the following classification of minimal compact Kähler surfaces S.

Classification of Surfaces

- $\kappa(S) = -1$, $S \cong \mathbb{P}^2$ (analogue of $\mathbb{P}^1$)
- $\kappa(S) = -1$, S is a ruled surface. There is a morphism $S \to B$ to a curve whose fibers are $\mathbb{P}^1$ (mixture of $\mathbb{P}^1$ and some other curve B)
- $\kappa(S) = 0$, S is one of the following surfaces: $K3$ surfaces, two-dimensional complex tori, Enriques surfaces, bielliptic surfaces (analogue of elliptic curves)
- $\kappa(S) = 1$, S is an elliptic surfaces. There is a morphism $S \to B$ to a curve whose smooth fibers are elliptic curves (mixture of elliptic curves and some other curve B)
- $\kappa(S) = 2$, S is a surface of general type (analogue of higher genus curves)
- In gross, S is either a mixture of curves (lower dimensional varieties), or one of the analogues of curves.

Analogues in Higher Dimensions

- One might tend to generalize surface classification methods to higher dimensions. This shall consists of two steps.

  (1) "Minimize" a given compact complex variety X.

  (2) Classify "minimal" compact complex varieties.

- The step (1) above is one of the motivation of higher dimensional birational geometry. Two varieties X and Y are called birationally equivalent if they are isomorphic after removing some proper subvarieties.
- For example, X is birational to a blow-up of it.
- One of the main ingredients of the step (2) is the abundance conjecture.

Minimal Model Programs

- In a surface X, a (-1)-curve C is a K_X-negative rational curve, that is, $C \cdot K_X < 0$. In higher dimensions, Mori's idea is to look at K_X-negative rational curve.
- For a projective manifold X, we define

$$\overline{\mathrm{NE}}(X) \subseteq H^{n-1,n-1}(X, \mathbb{C})$$

as the closed convex cone generated by curves, and

$$\mathrm{Nef}(X) \subseteq H^{1,1}(X, \mathbb{C})$$

as the closed convex cone generated by ample line bundles (divisors).

- The cones $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$ are dual to each other (Kleiman 1966). That is, a line bundle L is nef iff $C \cdot L \geqslant 0$ for any curve C.

Minimal Model Programs

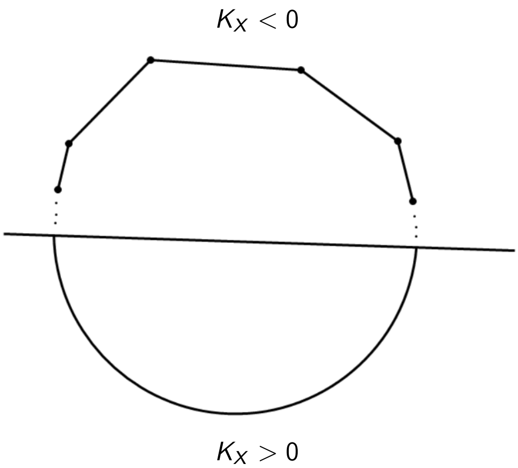
- Mori (1979) proved the so-called *bend-and-break* theorem, which first reveals the structure of $\overline{\mathrm{NE}}(X)$ (with the use of reduction modulo p method). Thanks to a sequence of work by other mathematicians, we have now the following Cone Theorem.

Cone Theorem (Mori, Kawamata, Reid, Shokurov, etc.)

Let X be a mildly singular projective variety.

1. There exist an at most countable set J of rational curves $C_j \subseteq X$ with $K_X \cdot C_j < 0$ such that
$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geqslant 0} + \sum_{j \in J} \mathbb{R}^+[C_j].$$
2. (Contraction Theorem.) Let $R = \mathbb{R}^+[C_j]$ be a K_X-negative extremal ray. Then there is a unique projective fibration $c_R : X \to Z$ such that a curve C is contracted by c_R if and only if the class of C is in R.

- A projective variety X (of dimension at least 3) is called *minimal* if K_X is nef, that is $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geqslant 0}$.



Section of some $\overline{\mathrm{NE}}(X)$

Minimal Model Programs

- Similarly to surface case, we wish to use the Contraction Theorem instead of Castelnuovo Theorem to minimize a given variety.
- Several difficulties raise in higher dimension. First of all, we need to deal with singular varieties.
- If an elementary contraction $c_R : X \to Z$ contracs only subvarieties of codimensions at least 2, we need to introduce a *flip* $X \dashrightarrow X^+$ to make the cone theorem work for X^+.
- Existence of flips has been proved for threefolds by Mori (1981), for fourfolds by Shokurov (2003) and in general by Birkar, and Hacon-Xu independently (2011).
- The remaining main difficulty is the non-existence of infinite sequence of flips. It is known for threefolds (Shokurov 1986, Kawamata 1992). A breakthrough in this problem has been made by Birkar-Cascini-Hacon-M^{c}Kernan (2010).

Minimal Model Programs

MMP Conjecture

Given a projective manifold X of dimension at least 3, after finitely many elementary operations, we can obtain a projective variety X', which is birationally equivalent to X, such that

- either there is a *Mori fibration* $f : X' \to Y$ with $\dim Y < \dim X'$ ($\nu(X') = -1$),
- or X' is a minimal variety ($\nu(X') \geqslant 0$).

Minimal Model Programs

- The invariant ν is called the numerical dimension. It is conjectured to be equal to the Kodaira dimension. We recover the three analogues of curves as follows.
- If $\nu(X') = -1$, and if the Mori fibration $f : X' \to Y$ is over a point Y, then X' is a *Fano* variety. This is an analogue of $\mathbb{P}^1$.
- If $\nu(X') = 0$ then X' is a *Calabi-Yau* variety. This is an analogue of elliptic curves.
- If $\nu(X') = \dim X'$, then X' is a variety of *general type*. This is an analogue of higher genus curves.
- As we will see, they are three building blogs in classifications of projective varieties.

Abundance Conjecture

- There should be two other types as in the surface case.
- Assume that $\nu(X') = -1$, and that the Mori fibration $f : X' \to Y$ is over a variety Y of positive dimension, then general fibers of f are Fano varieties of smaller dimensions. This case is an analogue of ruled surfaces, and can be regarded as a mixture of Fano varieties and some other lower dimensional variety Y.
- If $0 < \nu(X') < \dim X'$, we have the following conjecture.

Abundance Conjecture

Let X be a mildly singular projective variety such that K_X is nef. Then $\nu(X) = \kappa(X)$ and K_X is semiample. That is, K_X induces a fibration $f : X \to Y$ with $\dim Y = \nu(X) = \kappa(X)$.

- The abundance conjecture is true if $\nu(X) = 0$ or $\dim X$. It is also proved for surfaces and threefolds (Miyaoka, Kawamata 1990s).

Abundance Conjecture

- Assume that the abundance conjecture is true. Assume that $0 < \nu(X') < \dim X'$. Then we have a fibration $f : X' \to Y$ as in the abundance conjecture. In this case, $\dim Y = \nu(X')$ and general fibers of f are Calabi-Yau varieties. This case is an analogue of elliptic surfaces with Kodaira dimension 1, and can be regarded as a mixture of Calabi-Yau varieties and some other lower dimensional variety.

- From the viewpoint of MMP conjecture and abundance conjecture, we should recover exactly five types of classifications, as for minimal surfaces. We note that these two conjectures are known to be true for threefolds.
- More precisely, if X is a projective manifold of dimension at least 3, the two conjecture propose that X is birational to a variety Z, which is classified as follows.

Classification Conjecture

- We first recall the classification for minimal surfaces.
- $\kappa(S) = -1$, $S \cong \mathbb{P}^2$ (analogue of $\mathbb{P}^1$)
- $\kappa(S) = -1$, S is a ruled surface. There is a morphism $S \to B$ to a curve whose fibers are $\mathbb{P}^1$ (mixture of $\mathbb{P}^1$ and some other curve B)
- $\kappa(S) = 0$, S is one of the following surfaces: $K3$ surfaces, two-dimensional complex tori, Enriques surfaces, bielliptic surfaces (analogue of elliptic curves)
- $\kappa(S) = 1$, S is an elliptic surfaces. There is a morphism $S \to B$ to a curve whose smooth fibers are elliptic curves (mixture of elliptic curves and some other curve B)
- $\kappa(S) = 2$, S is a surface of general type (analogue of higher genus curves)

Classification Conjecture

- $\nu(Z) = -1$, Z is a Fano variety (analogue of $\mathbb{P}^1$)
- $\nu(S) = -1$, Z is a Mori fiber space. There is a Mori fibration $Z \to B$ to a variety B, whose general fibers are Fano varieties (mixture of Fano varieties and some other varieties B of smaller dimensions)
- $\nu(Z) = 0$, Z is a Calabi-Yau variety (analogue of elliptic curves)
- $0 < \nu(Z) < \dim Z$. There is a morphism $Z \to B$ to a variety whose general fibers are Calabi-Yau varieties (mixture of Calabi-Yau varieties and some variety B of smaller dimensions)
- $\nu(Z) = \dim Z$, Z is a variety of general type (analogue of higher genus curves)
- This classification allows us to study varieties by induction on dimensions.

MMP for Kähler Threefolds

- A compact complex manifold X is projective if and only if there is a rational Kähler form, that is, a Kähler form whose class is in $H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Q})$ (Kodaira 1954).
- There might be much fewer subvarieties in a Kähler manifold. For example, there are Kähler manifolds which do not have proper subvariety other than points. There are simple Kähler manifolds.
- **Kodaira problem.** Let X be a compact Kähler manifold. Can we approximate X by projective manifolds?
  - The answer is positive if $\dim X = 2$, after Kodaira.
  - There are negative examples if $\dim X \geqslant 4$, after Voisin.
  - Hsueh-Yung LIN recently give a positive answer for $\dim X = 3$.

- MMP for compact Kähler threefolds was initiated by Höring-Peternell. Following their ideas, the MMP for compact Kähler threefolds is now established, after Campana, Das, Hacon, Höring, Ou, Peternell...

MMP for Kähler threefolds

Given a compact Kähler threefold X, after finitely many elementary operations, we can obtain a compact Kähler variety X', which is bimeromorphic to X, such that

- either there is a *Mori fibration* $f : X' \to Y$ with $\dim Y < \dim X'$ ($\nu(X') = -1$),
- or X' is a minimal variety ($\nu(X') \geqslant 0$).

MMP for Kähler Threefolds

- Campana-Höring-Peternell (2016) announced the abundance conjecture for Kähler threefold. That is, if X is a compact Kähler threefold with mild singularities such that K_X is nef, then K_X is semiample.
- The proof of the abundance conjecture consists of two steps. The first step is to show that $\kappa(X) \geqslant 0$, that is $h^0(X, mK_X) > 0$ for some $m > 0$ sufficiently divisible. This is called the non-vanishing theorem.
- In the second step, we discuss according to the numerical dimension $\nu(X)$. The difficult cases are those when $\nu(X) = 1, 2$. Thanks to Kawamta, we only need to show that $\kappa(X) > 0$ in these cases.
- If $\nu(X) = 1$, then one apply deformation theory to conclude. The proof of the Kähler case is the same as the projective case.
- If $\nu(X) = 2$, then we also want to adapt the projective method to the Kähler case. But there is a main difficulty, which is the lack of Bogomolov-Gieseker type inequality for singular compact Kähler varieties.

Proof for $\nu(X) = 2$.

- We need to prove that $\kappa(X) > 0$. The idea is to use Riemann-Roch Theorem to show that $h^0(X, mK_X)$ grows at least linearly on m.

- We recall the RR Theorem. Let V be a compact smooth threefold, and L a line bundle on V. Then the Euler characteristic satisfies

$$\chi(V, mL) = h^0(V, mL) - h^1(V, mL) + h^2(V, mL) - h^3(V, mL)$$

$$\chi(V, mL) = \frac{L^3}{6} \cdot m^3 - \frac{K_V \cdot L^2}{4} \cdot m^2 + \frac{(K_V^2 + c_2(V)) \cdot L}{12} \cdot m + \chi(V).$$

- A simple case: X is smooth. We let $V = X$, $L = K_X$ in the previous RR formula. The condition $\nu(X) = 2$ implies that $L^3 = K_V \cdot L^2 = K_V^2 \cdot L = 0$. Then we get

$$\chi(X, mK_X) = \chi(V, mL) = \frac{c_2(V) \cdot L}{12} \cdot m + \chi(V).$$

- Since V is smooth and L is nef, after Enoki, BG inequality and Miyaoka, we deduce that $c_2(V)$ is pseudoeffective. Thus $c_2(V) \cdot L \geqslant 0$.

Proof for $\nu(X) = 2$.

- The difficulty arises when X is not smooth. We may let V be a desingularization of X. But $K_V^2 \cdot L < 0$ in general in the linear term
$$\frac{(K_V^2 + c_2(V)) \cdot L}{12} \cdot m$$

- We manage to show that $(K_V^2 + c_2(V)) \cdot L \geqslant \hat{c}_2(X) \cdot L$, where $\hat{c}_2$ is the $\mathbb{Q}$-Chern class, noting that X is klt and hence has quotient singularities in codimension 2.

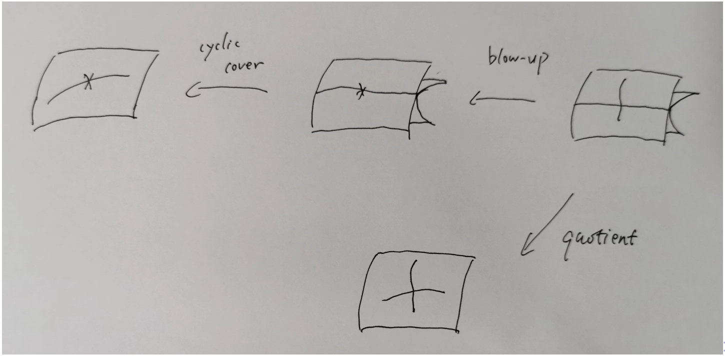
- A straightforward idea is to prove that $\hat{c}_2(X)$ is positive. Hence it is conjectured that BG-inequality holds for klt spaces: If (X, ω) is a compact Kähler variety with klt singulariites and $\mathcal{E}$ is stable reflexive coherent sheaf of rank r on X, then
$$\left(\hat{c}_2(\mathcal{E}) - \frac{r-1}{2r}\hat{c}_1(\mathcal{E})^2\right)\omega^{n-2} \geqslant 0.$$

- This conjecture is true when X has quotient singularities only, after Faulk (2019).

Proof for $\nu(X) = 2$.

- Solution (Das-Ou): We construct a bimeromorphic model $Y \to X$ so that Y has quotient singularities and $K_Y^2 \cdot L = 0$, $\hat{c}_2(Y) \cdot L \geqslant 0$, and conclude with Faulk's theorem.

- Fact: (X, Δ) a reduced dlt pair. Then, in codimension 2, for $x \in X$, either the pair is log smooth, or analytic locally $(X, \Delta) \cong \mathbb{C}^{n-2} \times (\mathbb{C}^2/\mathbb{Z}_m, E/\mathbb{Z}_m)$. The key point is that the quotient index is the same as the Cartier index of Δ.



Thank you !